No collaboration or aids are allowed. Prove every statement. Feel free to cite standard facts without proof, but clearly state the results you are using. If you write a partial solution, clearly indicate where the gaps are.

Problem 1 Let $M_2(\mathbb{Q})$ be the ring of 2×2 matrices with rational entries. Let R be the set of matrices in $M_2(\mathbb{Q})$ that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(1) Prove that R is a subring of $M_2(\mathbb{Q})$.
(2) Prove that R is isomorphic to the ring $\mathbb{Q}[x]/(x^2)$.

Problem 2 Let G be the group $\mathbb{Z}^3$, and let H be the subgroup of G generated by $(4, 6, 4)$, $(0, 6, -2)$, and $(4, 6, 8)$. Prove that the group G/H is finite and determine its order.

Problem 3 Let p be a prime and d a positive integer. Prove that the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ has a subfield F with degree $[F : \mathbb{Q}] = d$ if and only if $p \equiv 1 \mod d$.

Problem 4 Recall that a square matrix A is *nilpotent* if $A^n = 0$ for some positive integer. Determine the number of nilpotent 5×5 complex matrices up to similarity.

Problem 5 For a group G, let $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ be the set of group homomorphisms $G \to \mathbb{Q}/\mathbb{Z}$. For all $\psi, \phi \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ define the function $\psi * \phi : G \to \mathbb{Q}/\mathbb{Z}$ via $(\psi * \phi)(g) := \psi(g) + \phi(g)$ for all $g \in G$.

(1) Show that $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is an abelian group under the multiplication $*$.
(2) Show that if G is finite and cyclic, then $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong G$.
(3) Show that for all groups G and H,

$$\mathrm{Hom}(G \times H, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \times \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z}).$$

(4) Suppose G is a finitely-generated group. Show that $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong G$ if and only if G is finite and abelian.

Problem 6 Let R be a UFD and let K be its fraction field. Show that R is *integrally closed*: if $x \in K$ satisfies a monic polynomial equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

with coefficients $a_1, \ldots, a_n$ in R, then $x \in R$.

Problem 7 Prove that a group of order 280 is not simple.

Problem 8 Let $y = x^3 + x^{-3}$ be an element of the field $\mathbb{C}(x)$. For every intermediate field $\mathbb{C}(x) \supseteq F \supseteq \mathbb{C}(y)$, find an element $\alpha \in \mathbb{C}(x)$ such that $F = \mathbb{C}(y, \alpha)$.

Problem 9 Let $\mathbb{F}_p$ be the field of p elements where p is a prime. Let $\mathrm{GL}_n(\mathbb{F}_p)$ be the group of invertible $n \times n$ matrices. Determine the order of $\mathrm{GL}_n(\mathbb{F}_p)$.

Problem 10 Suppose F is a finite field, X is a set, and R is the ring of functions from X to F. Show that R is Noetherian if and only if X is finite.